Separable Statistics and Multivariate Linear Cryptanalysis

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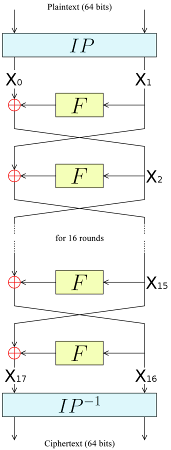
Vector of Internal Bits from Cipher



We define

$$A = (X_{16}[24, 18, 7, 29], X_{15}[16, 15, 14, 13, 12, 11], X_2[24, 18, 7, 29]).$$

The probability distribution of A depends on somme 7-bit $\tilde{k}$. We know (approximately) the probability distribution of A:

$$p(k) = (p_0, \dots, p_{2^{14}-1}),$$

where

$$p_i = \mathbf{Pr}\left(A = i \;\middle|\; \tilde{k} = k\right).$$

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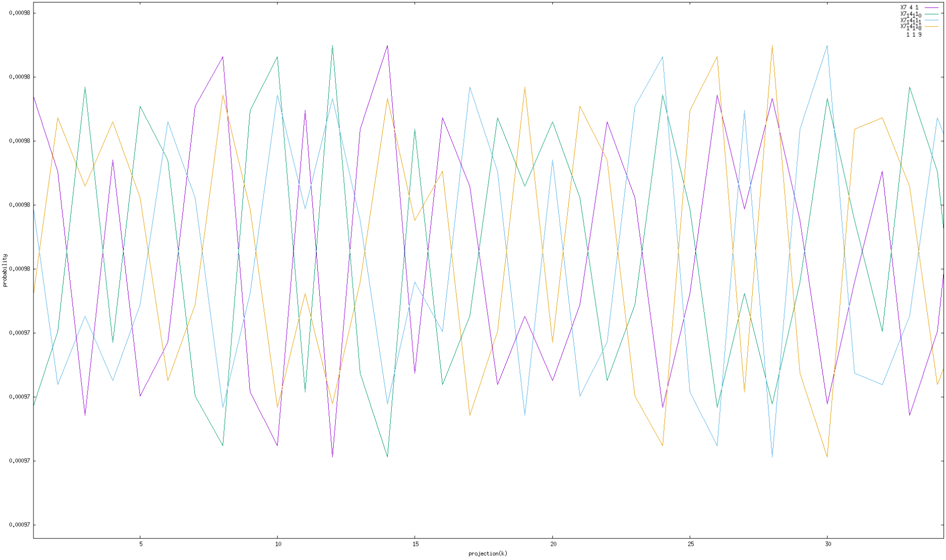
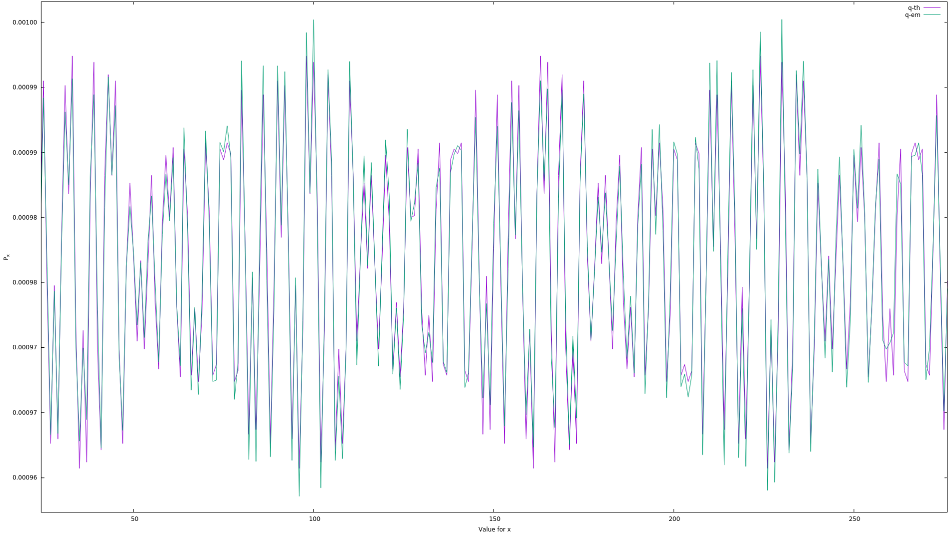
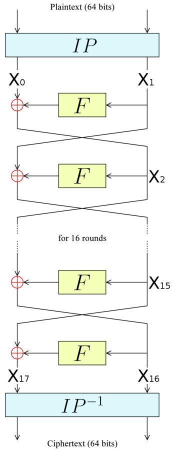

0,00098
0,00098
0,00098
0,00098
0,00098
0,00097
0,00097
0,00097
0,00097
5
10
15
20
25
30
projection(k)
probability
X7 4 1
X714110
X714111
X714118
1 1 9


q-th
q-em
0.00100
0.00099
0.00099
0.00098
0.00098
0.00097
0.00097
0.00096
P x
50
100
150
200
250
Value for x



$$A = (X_{16}[24, 18, 7, 29], X_{15}[16, 15, 14, 13, 12, 11], X_2[24, 18, 7, 29]).$$

We want to use A in a known plaintext attack on DES but X_2 and X_{15} is not part of the plaintext or ciphertext. We can, however, compute the relevant bits in X_2 and X_{15} from X_0, X_1, X_{16}, X_{17} and some 42-bit $\bar{k}$.

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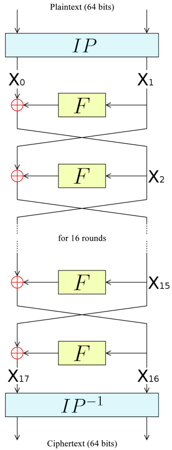

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Problem

$k \cup \bar{k} = 45$. We want time and data complexity to be $< 2^{43}$. Using the above vector in multivariate linear cryptanalysis [Hermelin et al.] would require that we rank 2^{45} key-candidates.

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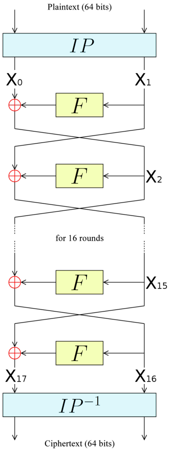

Instead of using A, we use 10-bit projections of A:

$$
\begin{aligned}
A^{(j)} &= (X_{16}[24, 18, 7, 29], X_{15}[a_j, b_j], X_2[24, 18, 7, 29]), \\
a_j, b_j &\in \{16, 15, 14, 13, 12, 11\}, \\
a_j &> b_j, \\
(a_j, b_j) &\neq (16, 11).
\end{aligned}
$$

There are 14 projections, $A^{(1)}, \ldots, A^{(14)}$. The probability distribution of $A^{(j)}$ can be computed from the probability distribution of A, and depends on some 2- or 3-bit $\tilde{k}^{(j)}$.

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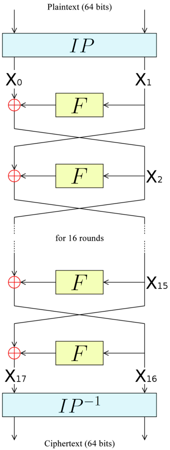
Computing $A^{(j)}$ from Observation



$$A^{(j)} = (X_{16}[24, 18, 7, 29], X_{15}[a_j, b_j], X_2[24, 18, 7, 29]).$$

Like before, we want to use $A^{(j)}$ in a known plaintext attack but X_2 and X_{15} is not part of the plaintext or ciphertext. We can, however, compute the relevant bits in X_2 and X_{15} from X_0, X_1, X_{16}, X_{17} and some 18-bit $\bar{k}^{(j)}$.

In total $A^{(j)}$ depends on 18-21 key-bits, denoted by $\mathtt{K}^{(j)} = \bar{k}^{(j)} \cup \tilde{k}^{(j)}$. 18 key-bits are needed to compute $A^{(j)}$ from a plaintext-ciphertext pair, and the distribution of $A^{(j)}$ depends on 2-3, possibly overlapping, key-bits.

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Random Vectors Based on Plaintext-Ciphertext Pairs

We observe n plaintext/ciphertext pairs all encrypted using the same key. We run over all plaintext-ciphertext pairs and compute the number of occurrences for each possible value of $A^{(j)}$ for all $\bar{k}^{(j)}$. We define a random vector (observation vector) for each $\bar{k}^{(j)}$

$$V^{(j)}(k) = (v_0^{(j)}, ..., v_{2^{10}-1}^{(j)}),$$

where $v_i^{(j)}$ is the number of times $A^{(j)} = i$ assuming $\bar{k}^{(j)} = k$.

$$V^{(j)}(k) = (v_0^{(j)}, \dots, v_{2^{10}-1}^{(j)})$$

is a random vector that follows multinomial distribution with n samples and some vector of probabilities, q. We have that:

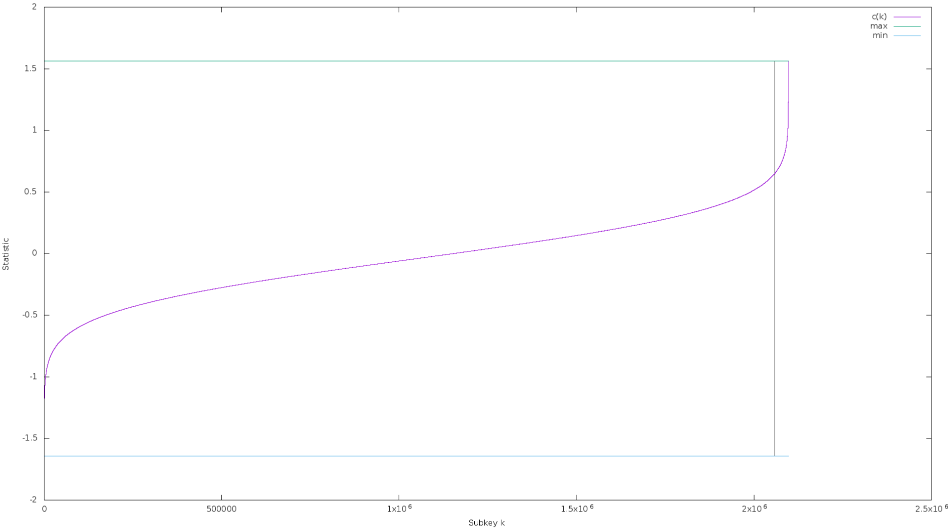
	guess of $\mathtt{K}^{(j)}$ correct	guess of $\mathtt{K}^{(j)}$ incorrect
$q =$	$p^{(j)}$	$(2^{-10}, \dots, 2^{-10})$
$E[v_i^{(j)}] =$	$n \times p_i^{(j)}$	$n \times 2^{-10}$
$Var[v_i^{(j)}] =$	$n \times p_i^{(j)} \times (1 - p_i^{(j)})$	$n \times 2^{-10} \times (1 - 2^{-10})$
$Cov[v_i^{(j)}, v_j^{(j)}] =$	$n \times p_i^{(j)} \times p_j^{(j)}$	$n \times 2^{-20}$

Separable Statistics

We compute the statistic $c^{(j)}(\mathtt{K}^{(j)})$ for all possible realisations of $\mathtt{K}^{(j)}$ and for all j. $c^{(j)}(\mathtt{K}^{(j)})$ is the log-likelihood-ratio of a correct guess of $\mathtt{K}^{(j)}$, over an incorrect guess of of $\mathtt{K}^{(j)}$.

$$c^{(j)}(\mathtt{K}^{(j)}) = log_2\left(\prod_i \left(\frac{p_i^{(j)}}{2^{-10}}\right)^{v_i^{(j)}}\right) = \sum_i v_i^{(j)} \times (log_2(p_i^{(j)}) + 10).$$

There are $< 14 \times 2^{21}$ possible realisations of $\mathtt{K}^{(j)}$ in total. Computing $c^{(j)}(\mathtt{K}^{(j)})$ for all of them can be done efficiently using fast Walsh-Hadamard Transform. The complexity is $O(2^{37})$ operations using $O(2^{28})$ memory.



c(k)
max
min
Statistic
Subkey k
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
500000
1x10 6
1.5x10 6
2x10 6
2.5x10 6

Symmetry in DES Cipher

Because of symmetry in DES it's trivial to duplicate all previous work using both A and A', which we assume are statistically independent.

$$A = (X_{16}[24, 18, 7, 29], X_{15}[16, 15, 14, 13, 12, 11], X_2[24, 18, 7, 29]) \ ,$$
$$A' = (X_1[24, 18, 7, 29], X_2[16, 15, 14, 13, 12, 11], X_{15}[24, 18, 7, 29]) \ .$$

We use 14 10-bit projections from each of them. $A^{(1)}, \ldots, A^{(14)}$ are projections of A and $A^{(15)}, \ldots, A^{(28)}$ are projections of A'. We now have 28 sub-keys, $\mathtt{K}^{(1)}, \ldots, \mathtt{K}^{(28)}$, and a statistic associated to each possible key value. That is, we have $< 28 \times 2^{21}$ different $c^{(j)}(\mathtt{K}^{(j)})$.

Let $\mathtt{K}$ be a 54-bit sub-key of the 56-bit key in DES. $\mathtt{K}$ is the union of $\mathtt{K}^{(1)}, \ldots, \mathtt{K}^{(28)}$. We want to use the previous statistics to find a good key candidate for $\mathtt{K}$. We define two separable statistics

$$C(\mathtt{K}) = \sum_{j=1}^{14} w_j \times c^{(j)}(\mathtt{K}^{(j)}) \qquad \text{and} \qquad C'(\mathtt{K}) = \sum_{j=15}^{28} w_j \times c^{(j)}(\mathtt{K}^{(j)}).$$

We built a search tree from the statistics $c^{(j)}(\mathtt{K}^{(j)})$ and designed an algorithm that goes through the tree to find 54-bit key candidates, $\mathtt{K}$. A key candidate is accepted if $C(\mathtt{K}) > z$ and $C'(\mathtt{K}) > z$ simultaneous, for some optimal weights w_j and a parameter z. The remaining 2 key-bits are brute forced for each key candidate.

Complexity and Probability of Success

The complexity of our attack is measured by n (number of plaintext-ciphertext pairs), the number of nodes visited while traversing the search tree and the number of encryptions to brute force the remaining 2 key-bits for all candidates.

$C(\mathtt{K})$ and $C'(\mathtt{K})$ are normally distributed. We choose z so that $n/4$ candidates for $\mathtt{K}$ are accepted. n encryptions is then performed.

The probability that our attack is successfull is the probability that $C(\mathtt{K}) > z$ and $C'(\mathtt{K}) > z$ for correct $\mathtt{K}$.

In particular, we set $n = 2^{41.8}$ and z so that the expected number of accepted candidates is $2^{39.8}$. Running the full attack returned $2^{39.46}$ candidates while visiting $2^{45.78}$ nodes in the search tree. Visiting one node is a simpler operation than one DES encryption, so the total time and data complexity is about $2^{41.8}$ encryptions. We are working on reducing the number of nodes visited.

Thanks

Questions?